Solution is independent of constraint value for minimal constraints.

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Abstract. I prove that for minimal constraints the solution to the normal equations is independent of the weight of the constraints. I also prove that the solution to the augmented normal equations is orthogonal to the constraints. Both of these are true in the exact case. Since the matrix inversion is done by computer roundoff error will spoil these results.

In solve we deal with normal matrices with some degeneracies. We usually remove these degeneracies by imposing constraints. In this note I show that if the constraints are minimal, i.e., we have only enough constraints to remove the degeneracies, then the solution we get is independent of the weight of the constraints.

We start by considering the normal equations:

$$NA = B \tag{1}$$

we wish to find a solution to these equations. For concreteness, we assume that N is $n \times n$ matrix. If N were not degenerate, we could invert this equation directly. So, assume that it is degenerate and has n_0 independent vanishing eigenvectors.

$$NZ_j = 0 \tag{2}$$

We want to modify equation 1 by appending to it a matrix of constraints:

$$(N + C)A = B \tag{3}$$

If we assume that C spans the null space of N, that is that

$$C \sum a_j Z_j \neq 0 \tag{4}$$

for all non-zero a_j then equation 3 can be inverted.

Now, any square symmetric matrix can be decomposed in the following fashion:

$$C = \begin{pmatrix} E \\ F \end{pmatrix} W \begin{pmatrix} E^T & F^T \end{pmatrix} \tag{5}$$

where E and W are square non-singular $m \times m$ matrices, and F is a rectangular $m \times (n - m)$ matrix. The dimension m is the rank of the matrix. If we demand that the columns of $\begin{pmatrix} E \\ F \end{pmatrix}$ be orthogonal then this decomposition is unique up to permutations. The matrix W is a weight matrix which determines how tightly the constraints are applied. If we further assume that

$$\mathrm{rank}(C) = n_0 \tag{6}$$

then the constraint matrix C is said to be minimal. In words we have done is imposed just enough constraints to invert the normal equation, but no more.

In the remainder of this note we will prove the following two propositions:

1. The solution A to equation 3 above is independent of W:

2. The solution is orthogonal to the constraint equation:

$$CA = 0 \tag{7}$$

Proof. We can always apply a similarity transform to get new normal equations which are equivalent to the old:

$$\left[S(N + C)S^{-1}\right] SA = SB \tag{8}$$

or

$$(N' + C')A' = B' \tag{9}$$

where

$$N' + C' = S(N + C)S^{-1} \tag{10}$$

$$A' = SA \tag{11}$$

$$B' = SB \tag{12}$$

The solution to the transformed normal equations is:

$$A' = [(N' + C')]^{-1} B' \tag{13}$$

It is well known that any symmetric matrix can be diagonalized by a similarity transform and there well established procedures for doing so. Suppose we multiply equation 3 by the transform that diagonalizes N: Without loss of generality we assume that the vanishing eigenvalues are the first n_0. By assumption

$$S(N + C)S^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + SCS^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + C' \tag{14}$$

here D is a diagonal matrix, and the second equality implicitly defines C':

$$C' = \begin{pmatrix} E' \\ F' \end{pmatrix} W \left(E'^T \quad F'^T \right) \tag{15}$$

where

$$\begin{pmatrix} E' \\ F' \end{pmatrix} = S \begin{pmatrix} E \\ F \end{pmatrix} . \tag{16}$$

Note also that from equation 1 we find

$$S^T B = B' = \begin{pmatrix} 0 \\ b' \end{pmatrix} . \tag{17}$$

Using the explicit decomposition of C' given in equation 15 one can show directly that

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + C' \right]^{-1} = \begin{pmatrix} E'^{T\,-1} \left(W^{-1} + F'^T D^{-1} F' \right) E'^{-1} & -E'^{T\,-1} F'^T D^{-1} \\ -D^{-1} F' E'^{-1} & D^{-1} \end{pmatrix} \tag{18}$$

which implies that

$$A' = [N' + C']^{-1} B' \tag{19}$$

$$= \left[\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + C' \right]^{-1} \begin{pmatrix} 0 \\ b' \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} -E'^{T\,-1} F'^T D^{-1} b' \\ D^{-1} b' \end{pmatrix} \tag{21}$$

Note first off that A', and hence A, is independent of W. This proves our first statement.

To prove the second, note that:

$$C'A' = 0$$

On the other hand,

$$C'A' = SCA \tag{22}$$

since S is non-singular, the vanishing of $C'A'$ implies the vanishing of CA, which proves our first statement.